Semi-Latin Squares and their Extensions

R. A. Bailey
University of St Andrews

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In the case that $k = 1$, we just get a Latin square of order n,
so we always assume that $k > 1$.

An example of a semi-Latin square

$A\ \alpha\ a$	$B\ \beta\ b$	$C\ \gamma\ c$	$D\ \delta\ d$	$E\ \varepsilon\ e$
$E\ \delta\ c$	$A\ \varepsilon\ d$	$B\ \alpha\ e$	$C\ \beta\ a$	$D\ \gamma\ b$
$D\ \beta\ e$	$E\ \gamma\ a$	$A\ \delta\ b$	$B\ \varepsilon\ c$	$C\ \alpha\ d$
$C\ \varepsilon\ b$	$D\ \alpha\ c$	$E\ \beta\ d$	$A\ \gamma\ e$	$B\ \delta\ a$
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$B \quad \gamma \quad d$	$C \quad \delta \quad e$	$D \quad \varepsilon \quad a$	$E \quad \alpha \quad b$	$A \quad \beta \quad c$

Here there are 5 rows, 5 columns, and 3 plots per cell.

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A α a	B β b	C γ c	D δ d	E ε e
E δ c	A ε d	B α e	C β a	D γ b
D β e	E γ a	A δ b	B ε c	C α d
C ε b	D α c	E β d	A γ e	B δ a
B γ d	C δ e	D ε a	E α b	A β c

Here there are 5 rows, 5 columns, and 3 plots per cell.

There are 15 treatments,
each occurring once in each row and once in each column.

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Once people started to include the cells in the data analysis, Frank Yates became enthusiastic about using semi-Latin squares.

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Subspace	dimension
Mean	1
Rows	$n-1$
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Cells within Rows and Columns	$(n-1)^2$
Plots within Cells	$n^2(k-1)$

General balance

Treatment contrasts are orthogonal to the first three strata,
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General balance is very useful if we want to combine treatment estimates using information from different strata.

Trojan squares

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Our previous example with $n = 5$ and $k = 3$ is Trojan.

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Theorem

If there is a Trojan square of size $(n \times n)/(n-1)$, then all of its inflations are A-optimal.

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Soicher proved that there exists a $(6 \times 6)/5s$ uniform semi-Latin square whenever $s \geq 2$.

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where each row is an East–West line of nk plots and
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This view led to widespread use of semi-Latin squares in agricultural research.

Extended semi-Latin squares: definition

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Now the n^2 cells of size k and the nk lines of size n both form systems of incomplete-blocks, so we have to allow for both of them in our design, randomization and data analysis.

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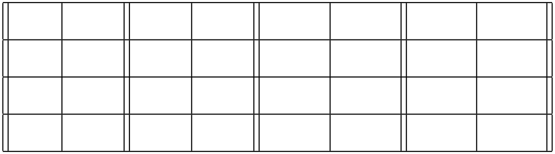

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Within each column, cells are crossed with lines.

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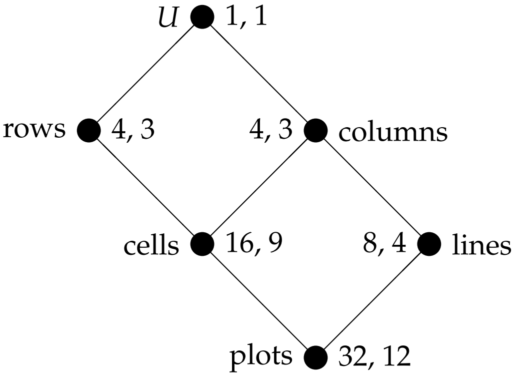

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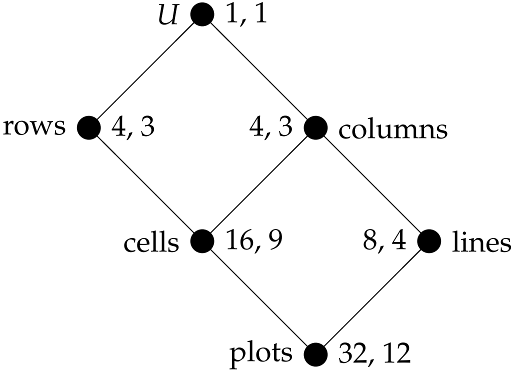

Here, U is the universal factor, which has a single level.
For each factor, the first number shows its number of levels and the second number shows its number of degrees of freedom.

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RAB's approach is to use patterns and combinatorics.
Since more information is lost to the block system with the smaller block size, it is better to start by finding the best design for that block system.

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In fact, Emlyn's software shows that this design is A-optimal.

Our paper

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Extended semi-Latin squares for use in field and glasshouse trials.
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doi: `10.1007/s13253-026-00729-6`